

Constraining Maximally Supersymmetric Membrane Actions

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ABSTRACT: We study the recent construction of maximally supersymmetric field theory Lagrangians in three spacetime dimensions that are based on algebras with a triple product. Assuming that the algebra has a positive definite metric compatible with the triple product, we prove that the only non-trivial examples are either the well known case based on a four dimensional algebra or direct sums thereof.

KEYWORDS: M-Theory, Supersymmetric gauge theory.

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1. Introduction

A better understanding of the three-dimensional superconformal field theory that arises on multiple membranes in flat space is an important outstanding issue in M-theory. Building on earlier work [1, 2], an interesting Lagrangian description of a maximally supersymmetric conformal field theory in three dimensions was constructed in [3, 4, 5] which has been further studied in [6] - [19]. The construction relies on an algebra with a skew triple product whose structure constants $f^{\mu_1\mu_2\mu_3}{}_\nu = f^{[\mu_1\mu_2\mu_3]}{}_\nu$ satisfy

$$f^{\mu_1\mu_2\mu_3}{}_\nu f^{\mu_4\mu_5\nu}{}_{\mu_6} = 3f^{\mu_4\mu_5[\mu_1}{}_\nu f^{\mu_2\mu_3]\nu}{}_{\mu_6} \quad (1.1)$$

or equivalently

$$f^{[\mu_1\mu_2\mu_3}{}_\nu f^{\mu_4]\mu_5\nu}{}_{\mu_6} = 0 . \quad (1.2)$$

The construction of the Lagrangian requires a compatible metric and, after raising an index on f using this metric, f is totally antisymmetric $f^{\mu_1\mu_2\mu_3\mu_4} = f^{[\mu_1\mu_2\mu_3\mu_4]}$. Since the metric appears in the kinetic terms of the Lagrangian, it is natural to demand that the metric is positive definite. In this case, after a suitable change of basis, we can assume that the metric is simply $\delta_{\mu\nu}$. The basic non-trivial solution [5] corresponds to a four dimensional algebra with $f^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}$. One can also consider direct sums of this basic example, but this simply leads to three-dimensional supersymmetric field theories which are non-interacting copies of the basic example.

We started this work by trying to construct additional solutions to (1.2) with totally antisymmetric f . However, as also noticed by others, obvious generalisations fail and simple computer searches are fruitless. It has also been shown [20] that in up to seven dimensions, a 4-form whose components satisfy (1.2) must be proportional

to dx^{1234} (in some appropriately chosen co-ordinates), and in eight dimensions, the solution is a linear combination dx^{1234} and dx^{5678} .

Here we will prove the general result, that all solutions of (1.2), in any dimension, can be written as a linear combination 4-forms, each of which is the wedge product of four 1-forms, which are all mutually orthogonal. This then proves conjectures made in [20] and [16].

Note added: Concurrent with the posting of this work to the ArXive, a proof of this result also appeared in [21]. After this paper was submitted for publication, we became aware of [22], which claims the same result using a different approach.

2. Analysis

We are interested in solutions to (1.2) for totally anti-symmetric and real f with indices raised and lowered using the metric $\delta_{\mu\nu}$. Let us assume that we have a $D+1$ dimensional algebra and write the indices as $\mu = (q, D+1)$ where $q = 1, \dots, D$. We can write

$$f = dx^{D+1} \wedge \psi + \phi \quad (2.1)$$

where ψ is a 3-form on \mathbb{R}^D , and ϕ is a 4-form on \mathbb{R}^D . We can demand that $\psi \neq 0$ (otherwise we end up in D dimensions). The constraint (1.2) is equivalent to

$$\phi^{[q_1 q_2 q_3] m} \phi^{q_4] q_5 q_6 m} + \psi^{[q_1 q_2 q_3} \psi^{q_4] q_5 q_6} = 0 \quad (2.2)$$

$$\phi^{[q_1 q_2 q_3} \psi^{q_4] q_5 m} = 0 \quad (2.3)$$

$$\phi^{q_1 q_2 q_3} \psi^{q_4 q_5 m} - 3\psi^{[q_1 q_2} \phi^{q_3] q_4 q_5 m} = 0 \quad (2.4)$$

$$\psi^{[q_1 q_2} \psi^{q_3] q_4 m} = 0 \quad (2.5)$$

where indices on ψ, ϕ are raised/lowered with δ_{mn} . Observe that (2.5) is the Jacobi identity. This identity implies that $\psi_{mn}{}^p$ are the structure constants of a Lie algebra \mathcal{L} . The Killing form of this Lie algebra has components

$$\kappa_{mn} = \psi_{m\ell}{}^p \psi_{np}{}^\ell. \quad (2.6)$$

As ψ is totally antisymmetric, note that κ is negative semi-definite. There are two possibilities: κ is non-degenerate and \mathcal{L} is semi-simple or κ is degenerate.

Suppose that \mathcal{L} is semi-simple. By making a $SO(D)$ rotation, one can diagonalize the Killing form and set

$$\kappa_{mn} = -\lambda_n \delta_{mn} \quad (2.7)$$

(no sum over n), and $\lambda_n > 0$ for all n .

On the other hand if κ is degenerate, then $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and \mathcal{L} is semi-simple. To see this we first note that $X^m \kappa_{mn} = 0$ for some non-zero vector X^n . Then it follows that

$$X^m X^n \psi_{mpq} \psi_n{}^{pq} = 0 \quad (2.8)$$

which implies that $X^n \psi_{npq} = 0$. Without loss of generality, one can make an $SO(D)$ rotation so that the only non-vanishing component of X^n is X^1 and then $\psi_{1mn} = 0$ for all m, n , and $\kappa_{1m} = 0$ for all m . By repeating this process in the directions $2, \dots, D$ one finds after a finite number of steps, either that $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and \mathcal{L}' is semi-simple, or $\psi = 0$ which we have assumed not to be the case.

We will analyse the two cases in turn, but we first establish some useful identities arising from (2.3)-(2.5) that are valid in both cases. We define $h = -\kappa$ i.e.

$$h_{mn} = \psi_{mab} \psi_n^{ab} . \quad (2.9)$$

First contract (2.3) with $\psi_{q_4 q_5 \ell}$ so that one obtains

$$\phi^{q_1 q_2 q_3 m} h_{m\ell} - \phi^{q_4 q_2 q_3 m} \psi^{q_5 q_1}_m \psi_{q_5 q_4 \ell} - \phi^{q_1 q_4 q_3 m} \psi^{q_5 q_2}_m \psi_{q_5 q_4 \ell} - \phi^{q_1 q_2 q_4 m} \psi^{q_5 q_3}_m \psi_{q_5 q_4 \ell} = 0 . \quad (2.10)$$

However, note that the Jacobi identity implies that

$$\phi^{q_4 q_2 q_3 m} \psi^{q_5 q_1}_m \psi_{q_5 q_4 \ell} = \frac{1}{2} \phi^{q_2 q_3 m n} \psi^{r q_1}_\ell \psi_{r m n} . \quad (2.11)$$

Using this identity one can rewrite (2.10) as

$$\phi^{q_1 q_2 q_3 m} h_{m\ell} - \frac{1}{2} \phi^{q_2 q_3 m n} \psi^{r q_1}_\ell \psi_{r m n} - \frac{1}{2} \phi^{q_3 q_1 m n} \psi^{r q_2}_\ell \psi_{r m n} - \frac{1}{2} \phi^{q_1 q_2 m n} \psi^{r q_3}_\ell \psi_{r m n} = 0 . \quad (2.12)$$

Also, contracting (2.3) with $\delta_{q_3 q_5}$ gives

$$\phi^{q_1 q_2 m n} \psi^{q_4}_{mn} + \phi^{q_2 q_4 m n} \psi^{q_1}_{mn} + \phi^{q_4 q_1 m n} \psi^{q_2}_{mn} = 0 . \quad (2.13)$$

Next, contract (2.4) with $\psi_{q_1 q_2 \ell}$ to obtain

$$-\phi^{q_3 q_4 q_5 m} h_{m\ell} + \phi^{q_1 q_2 q_3 m} \psi_{q_1 q_2 \ell} \psi^{q_4 q_5}_m - 2\phi^{q_2 q_4 q_5 m} \psi^{q_3 q_1}_m \psi_{q_1 q_2 \ell} = 0 . \quad (2.14)$$

This can be rewritten (using (2.11) to simplify the last term) as

$$-\phi^{q_1 q_2 q_3 m} h_{m\ell} + \phi^{m n q_1 r} \psi_{m n \ell} \psi^{q_2 q_3 r} + \phi^{q_2 q_3 m n} \psi^{r q_1}_\ell \psi_{r m n} = 0 . \quad (2.15)$$

On contracting this expression with $\delta_{q_1 q_3}$, the first and the third term vanish (the third term vanishes as a consequence of the Jacobi identity), and we find

$$\phi^{n_1 n_2 m_1 m_2} \psi_{n_1 n_2 \ell} \psi_{m_1 m_2 r} = 0 . \quad (2.16)$$

Next, contract (2.15) with $\psi_{q_2 q_3 s}$. The last term vanishes as a consequence of (2.16), and we obtain

$$-\phi^{m n q r} h_{r\ell} \psi_{m n s} + \phi^{m n q r} h_{rs} \psi_{m n \ell} = 0 . \quad (2.17)$$

2.1 Solutions when \mathcal{L} is semi-simple

We now assume that \mathcal{L} is semi-simple. As we have already observed, we can make a rotation and work in a basis for which

$$h_{mn} = \lambda_n \delta_{mn} \quad (2.18)$$

(no sum over n), with $\lambda_n > 0$ for all n .

Then (2.17) implies

$$-\phi^{mnq} \ell \lambda_\ell \psi_{mns} + \phi^{mnq} s \lambda_s \psi_{mn\ell} = 0 \quad (2.19)$$

with no sum over ℓ or s . On substituting this expression back into (2.13) we obtain

$$(\lambda_{q_4} - \lambda_{q_1} - \lambda_{q_2}) \phi^{q_1 q_2 mn} \psi^{q_4}_{mn} = 0 \quad (2.20)$$

(no sum on q_1, q_2, q_4). Hence $\phi^{q_1 q_2 mn} \psi^{q_4}_{mn} = 0$, or $\lambda_{q_4} - \lambda_{q_1} - \lambda_{q_2} = 0$ for some choice of q_1, q_2, q_4 . Now, it is not possible to have $\lambda_{q_4} - \lambda_{q_1} - \lambda_{q_2} = \lambda_{q_1} - \lambda_{q_2} - \lambda_{q_4} = \lambda_{q_2} - \lambda_{q_1} - \lambda_{q_4} = 0$ simultaneously. Hence, at least one of $\phi^{q_1 q_2 mn} \psi^{q_4}_{mn}$, $\phi^{q_1 q_4 mn} \psi^{q_2}_{mn}$, $\phi^{q_2 q_4 mn} \psi^{q_1}_{mn}$ must vanish. However, (2.19) then implies that *all* these terms vanish. Hence we conclude that

$$\phi^{q_1 q_2 mn} \psi^{q_4}_{mn} = 0 \quad (2.21)$$

for all q_1, q_2, q_4 . Finally, on substituting (2.21) back into (2.12), the last three terms are constrained to vanish, hence

$$\phi_{q_1 q_2 q_3 q_4} = 0. \quad (2.22)$$

Now consider (2.2). This implies that

$$\psi^{[q_1 q_2 q_3] \psi^{q_4] q_5 q_6} = 0 \quad (2.23)$$

which implies (see e.g. [20]) that ψ is simple i.e. it can be written as the wedge product of three one forms. Hence one can chose a basis for which

$$\psi = \lambda dx^1 \wedge dx^2 \wedge dx^3. \quad (2.24)$$

Furthermore, as \mathcal{L} is compact, this implies that \mathcal{L} must be 3-dimensional i.e. $\mathcal{L} = su(2)$. We have thus recovered the basic four-dimensional case with $f^{\mu_1 \mu_2 \mu_3 \mu_4} = \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$.

2.2 Solutions when \mathcal{L} is not semi-simple

Set $\mathcal{L} = u(1)^p \oplus \mathcal{L}'$ where $p > 0$ and \mathcal{L}' is semi-simple. It will be useful to split the indices m into “semi-simple” directions \hat{m} and “ $u(1)$ ” directions A , so $m = (\hat{m}, A)$.

Note that $\psi_{Amn} = 0$ for all m, n , and $h_{Am} = 0$ for all m , but $h_{\hat{m}\hat{n}} = \lambda_{\hat{n}}\delta_{\hat{m}\hat{n}}$ (no sum on \hat{n}). Recall the identity (2.12). Setting $q_1 = A$, $q_2 = B$, $q_3 = C$ one finds

$$\phi_{ABC\hat{m}} = 0 . \quad (2.25)$$

Also, setting $q_1 = A$, $q_2 = B$, $q_3 = \hat{m}$ one finds

$$\phi^{AB\hat{m}\hat{s}}h_{\hat{s}\hat{\ell}} - \frac{1}{2}\phi^{AB\hat{p}\hat{q}}\psi_{\hat{\ell}}^{\hat{s}\hat{m}}\psi_{\hat{s}\hat{p}\hat{q}} = 0 . \quad (2.26)$$

However, (2.13) implies that

$$\phi^{AB\hat{p}\hat{q}}\psi_{\hat{s}\hat{p}\hat{q}} = 0 \quad (2.27)$$

and so on substituting this back into (2.26) one finds

$$\phi_{AB\hat{m}\hat{n}} = 0 . \quad (2.28)$$

Returning to the general conditions (2.3), (2.4) and (2.5) with all free indices hatted, we can follow the same steps in the last subsection to conclude that

$$\phi_{\hat{m}\hat{n}\hat{p}\hat{q}} = 0 . \quad (2.29)$$

Thus the only non-zero components of ϕ are of the form $\phi^{A\hat{q}_1\hat{q}_2\hat{q}_3}$ and ϕ^{ABCD} .

Considering other indices in (2.3), (2.4) and (2.5) we conclude that

$$\psi^{[\hat{q}_1\hat{q}_2}_{\hat{m}}\psi^{\hat{q}_3]\hat{q}_4\hat{m}} = 0 \quad (2.30)$$

$$\phi^{A\hat{q}_1\hat{q}_2}_{\hat{m}}\psi^{\hat{q}_3\hat{q}_4\hat{m}} = \phi^{A\hat{q}_3\hat{q}_4}_{\hat{m}}\psi^{\hat{q}_1\hat{q}_2\hat{m}} \quad (2.31)$$

$$\phi^{A\hat{q}_1[\hat{q}_2}_{\hat{m}}\psi^{\hat{q}_3\hat{q}_4]\hat{m}} = 0 . \quad (2.32)$$

From (2.2) we also get

$$\phi^{[A_1A_2A_3}_B\phi^{A_4]A_5A_6B} = 0 \quad (2.33)$$

$$\phi^{\hat{q}_1\hat{q}_2\hat{q}_3}_B\phi^{A_1A_2A_3B} = 0 \quad (2.34)$$

$$\phi^{A[\hat{q}_1\hat{q}_2}_{\hat{m}}\phi^{\hat{q}_3]\hat{q}_4B\hat{m}} = 0 \quad (2.35)$$

$$\phi^{\hat{q}_1\hat{q}_2}_{\hat{m}}[\phi_{A_1}\phi_{A_2}]^{\hat{q}_3\hat{q}_4\hat{m}} = 0 \quad (2.36)$$

$$\psi^{[\hat{q}_1\hat{q}_2\hat{q}_3}\psi^{\hat{q}_4]\hat{q}_5\hat{q}_6} + \phi^{[\hat{q}_1\hat{q}_2\hat{q}_3}_A\phi^{\hat{q}_4]\hat{q}_5\hat{q}_6A} = 0 . \quad (2.37)$$

To proceed with the analysis, it is convenient to define the matrices T^A by

$$(T^A)_{\hat{m}}^{\hat{n}} = \phi^{A\hat{q}_1\hat{q}_2\hat{n}}\psi_{\hat{q}_1\hat{q}_2\hat{m}} . \quad (2.38)$$

On contracting (2.31) with $\delta_{\hat{q}_2\hat{q}_4}$, we observe that T^A are all symmetric matrices. Furthermore, on contracting (2.36) with $\delta_{\hat{q}_2\hat{q}_4}$ and making use of (2.31), it is straightforward to show that the matrices T^A commute with each other. Also, (2.31) implies that the T^A commute with h .

Next, note that the Jacobi identity (2.30) implies that

$$(T^A)_{\hat{m}\hat{\ell}}\psi_{\hat{p}\hat{q}}^{\hat{\ell}} = \phi^{A\hat{s}\hat{t}}_{\hat{m}}\psi_{\hat{s}\hat{t}\hat{\ell}}\psi_{\hat{p}\hat{q}}^{\hat{\ell}} = -2\phi^{A\hat{s}\hat{t}}_{\hat{m}}\psi_{\hat{s}\hat{p}\hat{\ell}}\psi_{\hat{q}\hat{t}}^{\hat{\ell}} \quad (2.39)$$

However, now using (2.31) and then the Jacobi identity again, we get

$$-2\phi^{A\hat{s}\hat{t}}_{\hat{m}}\psi_{\hat{s}\hat{p}\hat{\ell}}\psi_{\hat{q}\hat{t}}^{\hat{\ell}} = -2\phi^{A\hat{s}}_{\hat{p}\hat{i}}\psi_{\hat{s}}^{\hat{t}}\psi_{\hat{m}}^{\hat{i}}\psi_{\hat{q}\hat{t}}^{\hat{\ell}} = -\phi^{A\hat{s}\hat{t}}_{\hat{p}}\psi_{\hat{s}\hat{t}\hat{\ell}}\psi_{\hat{m}\hat{q}}^{\hat{\ell}}. \quad (2.40)$$

Thus

$$(T^A)_{\hat{m}\hat{\ell}}\psi_{\hat{p}\hat{q}}^{\hat{\ell}} = -(T^A)_{\hat{p}\hat{\ell}}\psi_{\hat{m}\hat{q}}^{\hat{\ell}}. \quad (2.41)$$

Next, decompose semi-simple $\mathcal{L}' = \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_m$ where \mathcal{L}_i are simple ideals such that $\mathcal{L}_i \perp \mathcal{L}_j$ (with respect to h), and $[\mathcal{L}_i, \mathcal{L}_j] = 0$ if $i \neq j$, and the restriction of the adjoint rep. to \mathcal{L}_i is irreducible; furthermore, $h|_{\mathcal{L}_i} = 2\mu_i^2 \mathbb{I}$ for $\mu_i \neq 0$. Contract (2.31) with $\psi_{\hat{q}_3\hat{q}_4\hat{\ell}}$ to obtain

$$\phi^{A\hat{q}_1\hat{q}_2\hat{m}}h_{\hat{m}\hat{\ell}} = \phi^{A\hat{q}_3\hat{q}_4\hat{m}}\psi_{\hat{r}\hat{\ell}\hat{q}_1}^{\hat{q}_1\hat{q}_2}\psi_{\hat{q}_3\hat{q}_4\hat{\ell}}. \quad (2.42)$$

Suppose that the indices \hat{q}_1, \hat{q}_2 lie in two different ideals $\mathcal{L}_i, \mathcal{L}_j$ for $i \neq j$. Then the RHS of the above expression vanishes, hence for these indices, $\phi_{A\hat{q}_1\hat{q}_2\hat{m}} = 0$, for all \hat{m} . Similarly, for these indices $(T^A)_{\hat{q}_1}^{\hat{q}_2} = \phi^{A\hat{r}\hat{\ell}\hat{q}_2}\psi_{\hat{r}\hat{\ell}\hat{q}_1} = 0$.

Consider T_i^A , the restriction of T^A to \mathcal{L}_i . Then (2.41) implies that T_i^A commutes with the restriction of the adjoint rep. to \mathcal{L}_i . However, as this restriction of the adjoint rep. is irreducible, it follows by Schur's Lemma that

$$T_i^A = \lambda_i^A \mathbb{I}. \quad (2.43)$$

As the T^A all commute, this can be achieved for all T^A .

Next, consider (2.37) with all \hat{q} indices restricted to \mathcal{L}_i . Contracting this expression with $\psi_{\hat{q}_1\hat{q}_2\hat{q}_3}\psi_{\hat{q}_5\hat{q}_6\hat{m}}$ gives

$$\left(\sum_A (\lambda_i^A)^2 + 4(\mu_i)^4 \right) \left(\dim \mathcal{L}_i - 3 \right) \delta_{\hat{m}}^{\hat{q}_4} = 0 \quad (2.44)$$

which implies that $\dim \mathcal{L}_i = 3$ for all i , so $\mathcal{L}_i = su(2)$. It follows that

$$\psi = \sum_i \mu_i \theta_i \quad (2.45)$$

with $\mu_i \neq 0$, where

$$\theta_i = dy_i^1 \wedge dy_i^2 \wedge dy_i^3 \quad (2.46)$$

If the \hat{q} indices are restricted to \mathcal{L}_i , since $\dim \mathcal{L}_i = 3$, $\phi_{A\hat{q}_1\hat{q}_2\hat{q}_3}$ must be proportional to θ_i . The proportionality constant can be fixed from (2.43) and we find

$$\phi_{A\hat{q}_1\hat{q}_2\hat{q}_3} = \frac{\lambda_i^A}{2\mu_i} (\theta_i)_{\hat{q}_1\hat{q}_2\hat{q}_3}. \quad (2.47)$$

It is convenient to re-define $\lambda_i^A = 2\mu_i\chi_i^A$, so that

$$f = dx^{d+1} \wedge \psi + \sum_{i,A} \chi_i^A dz^A \wedge \theta_i + \Phi \quad (2.48)$$

where Φ lies entirely in the $u(1)$ directions, whose directions we have denoted by z^A . The remaining content of (2.37) is obtained by restricting the indices $\hat{q}_1, \hat{q}_2, \hat{q}_3$ to \mathcal{L}_i , and $\hat{q}_4, \hat{q}_5, \hat{q}_6$ to \mathcal{L}_j for $i \neq j$; we find

$$\mu_i\mu_j + \sum_A \chi_i^A \chi_j^A = 0. \quad (2.49)$$

Note that the form Φ satisfies the quadratic constraint (2.33), whereas (2.34) is equivalent to

$$\chi_i^A \Phi_{AMNP} = 0 \quad (2.50)$$

for all i .

There are then two cases to consider. In the first case, $\chi_i^A = 0$ for all A, i . Then (2.49) implies that $\mathcal{L}' = su(2)$, and hence

$$f = \mu_1 dx^{d+1} \wedge dy_1^1 \wedge dy_1^2 \wedge dy_1^3 + \Phi \quad (2.51)$$

where Φ has no components in the $x^{d+1}, y_1^1, y_1^2, y_1^3$ directions.

In the second case, there exists some A, i with $\chi_i^A \neq 0$. Without loss of generality, take $i = 1$. By making an $SO(p)$ rotation entirely in the $u(1)$ directions, without loss of generality set

$$\chi_1^1 = \tau, \quad \chi_1^A = 0 \quad \text{if } A > 1 \quad (2.52)$$

where $\tau \neq 0$. Then, if $j \neq 1$, (2.49) implies that

$$\chi_j^1 = -\frac{\mu_1}{\tau} \mu_j. \quad (2.53)$$

Substituting these constraints back into (2.48), and rearranging the terms, one finds

$$\begin{aligned} f &= (\mu_1 dx^{d+1} + \tau dz^1) \wedge \theta_1 + \tau^{-1}(\tau dx^{d+1} - \mu_1 dz^1) \wedge \sum_{j>1} \mu_j \theta_j \\ &+ \sum_{j>1, A>1} \chi_j^A dz^A \wedge \theta_j + \Phi. \end{aligned} \quad (2.54)$$

Writing

$$\begin{aligned} f_1 &= (\mu_1 dx^{d+1} + \tau dz^1) \wedge \theta_1 \\ \tilde{f} &= \tau^{-1}(\tau dx^{d+1} - \mu_1 dz^1) \wedge \sum_{j>1} \mu_j \theta_j + \sum_{j>1, A>1} \chi_j^A dz^A \wedge \theta_j + \Phi \end{aligned} \quad (2.55)$$

we have found $f = f_1 + \tilde{f}$ where, as a consequence of (2.50) and (2.52), it follows that Φ has no components in the z^1 direction.

So, in both cases, we have the decomposition

$$f = f_1 + \tilde{f} \quad (2.56)$$

where f_1 is a simple 4-form, and f_1, \tilde{f} are totally orthogonal i.e. $f_1^{\mu_1\mu_2\mu_3\nu} \tilde{f}_{\mu_4\mu_5\mu_6\nu} = 0$.

Having obtained this result, it is straightforward to prove that if such an f satisfies (1.2), then

$$f = \sum_{s=1}^N f_s \quad (2.57)$$

where f_s are totally orthogonal simple 4-forms. The proof proceeds by induction on the spacetime dimension D ($D \geq 4$). The result is clearly true for $D = 4$. Suppose it is true for $4 \leq D \leq d$. Suppose that $D = d + 1$. Then by the previous reasoning, one has the decomposition $f = f_1 + \tilde{f}$, where f_1 is a simple 4-form, and f_1, \tilde{f} are totally orthogonal. It follows that \tilde{f} must satisfy (1.2). Then either $\tilde{f} = 0$ and we are done, or \tilde{f} is a nonzero 4-form in dimension $d - 3$, in which case it follows that one can decompose \tilde{f} into a finite sum of orthogonal simple 4-forms, each of which is also orthogonal to f_1 .

Hence we conclude that the decomposition (2.57) holds for all 4-forms f satisfying (1.2).

3. Discussion

Given the results presented here, the maximally supersymmetric field theory Lagrangian based on the four-dimensional algebra with $f^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_1\mu_2\mu_3\mu_4}$ is rather enigmatic. If it is not to be an isolated curiosity, the assumptions going into the general constructions of [3, 4, 5] need to be relaxed. One possibility is to relax the condition that the metric living on the algebra is positive definite and some discussion recently appeared in [16]. A different possibility is to not demand a Lagrangian description, but to work instead at the level of the field equations and this was recently discussed in [15]. Another possibility, which also does not use totally antisymmetric structure constants, was considered in [12].

Acknowledgments

We would like to thank Per Kraus for discussions. JPG is supported by an EPSRC Senior Fellowship and a Royal Society Wolfson Award.

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